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Flat Connection Contribution to Topology Changing Amplitudes in an Ensemble of Seifert Fibered Homology Spheres

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The Fintushel–Stern pseudofree orbifolds are exploited to construct wave functions of universes created as a result of the interaction of cones on lens spaces. We also study the problem of the definition of topology changing amplitudes for tunneling topology changes, described by cobordisms with Seifert fibered homology sphere boundaries. It is demonstrated that such topology changes are accompanied by creation or annihilation of the lens spaces. The topology-changing amplitude calculations are carried out in the stationary phase approximation for Kodama wave functions. In this approximation the changing amplitudes factorize and they are expressed by means of Chern–Simons invariants of flat connections over Seifert fibered homology spheres and lens spaces.

1. INTRODUCTION

The problem of changing topology has been discussed from different points of view in both classical and quantum gravity (Horowitz, 1991). From the conceptual point of view a future quantum theory of gravitation is dependent on a resolution of this problem. In addition, the investigation of topology changes may bring a new approach to the early cosmology of the universe (Hawking, 1988; Weinberg, 1989) and in particular a resolution of the cosmological constant question (Coleman, 1988; Klebanov *et al.*, 1989), as well as fixing other fundamental constants of nature (Preskill, 1989; Weinberg, 1989). But the complicated dynamical structure of Einstein's field equation and difficulties in the 3- and 4-dimensional topologies does not allow one to achieve in the (3 + 1)D gravity at least the same level of understanding of the topology change problem in the (2 + 1)-dimensional case (Witten,

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1988, 1989). However progress has been achieved recently in this direction. It is based on the development of a new approach to low-dimensional space topology established by gauge field theory [see, for example, reviews by Freed and Uhlenbeck (1984), Okonek (1991), and Rozansky (1995)]. On the other hand, progress has been initiated by the ideas of Ashtekar *et al.* [see for review Ashtekar (1991)] of an alternative representation for Hamiltonian general relativity (GR). In the same way as in (2 + 1)D gravity (Witten, 1988; Ashtekar *et al.*, 1989), Ashtekar's method permits one to formulate GR in connection (Jacobson and Smolin, 1988) and loop (Rovelli and Smolin, 1990) representations. But this approach gives a possibility of defining a complete set of dynamical variables for gravity only in a (2 + 1)-dimensional toy model. A reason, in particular, is that the solutions of (2 + 1)D Einstein equations are flat Poincaré connections. These are characterized by their holonomy along generators of the fundamental group $\pi_1(\Sigma^2)$ of the spacelike 2D section Σ^2 of the 3-dimensional space-time. In the case of topology changes in 4-dimensional space-time, we also can restrict our consideration to the equivalence classes of flat connections on 3D sections of 4-dimensional cobordism, which describes the topology change. Then the loop variables (Rovelli and Smolin, 1990; Ashtekar *et al.*, 1992) are defined uniquely on homotopy equivalence classes of loops.

We make one more observation of the (2 + 1)D toy model. It was demonstrated (Martin, 1989; Furuta and Steer, 1992) that in 3-dimensional gravity it is possible to introduce a nontrivial dynamics by allowing conical singularities representing point particles. Analogously it might be useful to enrich the (3 + 1)D theory by admitting one-dimensional exceptional orbits (fibers) corresponding to string structures. It is known (Eisenbud and Neumann, 1985) that the Seifert fibers and the Seifert links provide an appropriate model of spacelike sections which have such exceptional orbits relative to the action of the group S^1 [$\simeq U(1)$]. Examples of spaces which admit the natural Seifert fibers are homology spheres and lens spaces (Scott, 1983). We consider precisely this type of manifold as admissible 3-dimensional sections of Euclidean space-time cobordisms describing topology changes. Such an ensemble of 3-dimensional manifolds makes it possible to describe an interlacement (splicing) of Seifert fibers as a result of topology change. It is possible to connect this phenomenon with a refinement (or simplification) of the structure of the universe as a consequence of different phase transformations, for example, a breakdown (or restoration) of symmetry, i.e., a change of a number of the fundamental interactions in the universe. In addition it would be attractive to interpret the topology changes in terms of elementary particle processes.

Section 2 gives basic definitions and notations.

In Section 3 elementary cobordisms of two types are constructed. They are used as constituent parts for formation of the various 4D spaces describing topology changes.

A procedure of sewing together elementary cobordisms is formalized in Section 4 by means of the well-known topology operation of "splicing."

Section 5 describes a naive method for transition from the Lorentzian region to the Euclidean domain and conversely, when the collection of homology spheres are boundaries of the Euclidean regions with a flat connection on them. The Ashtekar canonical variables are used for this description.

In Section 6 the topology changing amplitudes are constructed in connection and loop representations. A stationary phase approximation is utilized. The basic assumption is that all physical quantum states are represented in the expansion form with respect to Kodama wave functions (Kodama, 1990).

2. BASIC NOTATIONS AND DEFINITIONS

The fundamental objects of our investigation are the lens spaces, Seifert fibered homology spheres (Sfh-spheres), and cobordisms with boundaries, which are different combinations of these spaces. For completeness, we recall the definitions and notations for these manifolds (see, for example, Rolfsen, 1976).

The lens space $L(a, b)$ $(a, b \in Z, a > 1)$ is a factor space of

$$S^3 = \{z, w| \, |z|^2 + |w|^2 = 1\}$$

with respect to the free action of the group $Z_a \subset S^1$; this action is defined by $(z, w) = (\zeta^b z, \zeta w)$, where $\zeta = \exp(2\pi i/a)$. Consequently, a fundamental group of $L(a, b)$ is

$$\pi_1(L(a, b)) \cong Z_a \tag{2.1}$$

It is natural to define Seifert fibered homology spheres $\Sigma(\underline{a}) \equiv \Sigma(a_1, \ldots, a_n)$ (Fintushel and Stern, 1985) beginning from the notion of a Seifert fibered manifold with invariants (a_i, b_i), $i = 1, \ldots, n$ $(a_i, b_i \in Z, a_i > 1;$ a_i, b_i are relatively prime for each i).

The Seifert fiber manifold is a 3-dimensional manifold $\Sigma(\underline{a})$ with the pseudofree S^1-action. The pseudofree S^1-action is a smooth action such that it is free except for finitely many exceptional orbits $S_1, \ldots, S_n$ with isotropy groups (stabilizers) $Z_{a_1}, \ldots, Z_{a_n}$, respectively, where $a_1, \ldots, a_n$ are pairwise relatively prime. Thus the Seifert fibered manifold $\Sigma(\underline{a})$ is endowed with the structure of S^1-fibering (Seifert structure), with base space (Fintushel–Stern pseudofree 2-dimensional orbifold)

$$\Sigma(\underline{a})/S^1 = S^2(a_1, \ldots, a_n) \equiv S^2(\underline{a}) \tag{2.2}$$

which is homeomorphic to the sphere S^2 (underlying Riemann surface).

The orbifold $S^2(\underline{a})$ possesses n exceptional cone points (with conical angles $2\pi/a_i$, $i = 1, \ldots, n$).

To the end of assigning the Seifert structure more explicitly, let us consider an n disjoint open tubular (torus) neighborhood

$$TS_1 = (S^1 \times D^2)_1, \ldots, TS_n = (S^1 \times D^2)_n$$

of exceptional fibers $S_1, \ldots, S_n$. Then by definition of a Seifert fiber, there exists a trivial S^1-fibering

$$\begin{gathered} p\colon\ \Sigma_0 \to F_0 \\ \Sigma_0 = \Sigma - TS_1 \sqcup \cdots \sqcup TS_n \\ F_0 = S^2 - D_1^2 \sqcup \cdots \sqcup D_n^2 \end{gathered} \tag{2.3}$$

(the symbol $\sqcup$ denotes disjoint union and the disc D_i^2 is a neighborhood of i's cone point).

If $R \subset \Sigma_0$ is a (cross) section, then $x_i = -\partial R \cap TS_i$ is a torus curve (link) $i = 1, \ldots, n$. If h is a typical orbit (fiber) of S^1-fibering p, then the curve $a_i x_i + b_i h$ is homologic to zero (it is a boundary of some surface) in TS_i. The fundamental group presentation of a Seifert fibered manifold is

$$\pi_1(\Sigma(\underline{a})) = \langle x_1, \ldots, x_n, h \,|\, hx_i h^{-1} x_i^{-1} = 1, x_i^{a_i} h^{b_i} = 1, x_1 \cdots x_n = 1 \rangle \tag{2.4}$$

The group S^1 acts freely along a typical fiber h, and in the tubular neighborhood $TS_i = (S^1 \times D^2)_i$ of the exceptional orbit S_i this S^1-action is

$$t \circ (s, z) = (t^{a_i} s, t^{\sigma_i} z)$$

where $\sigma_i = a/a_i$, $a = a_1 \cdots a_n$; $(s, z) \in S^1 \times D^2$, $S^1 \subset \mathbb{C}$, $D^2 \subset \mathbb{C}$.

An oriented Seifert fibered manifold $\Sigma(\underline{a}) \equiv \Sigma(a_1, \ldots, a_n)$ with invariants $\{(a_i, b_i) \,|\, i = 1, \ldots, n\}$ satisfying

$$\sum_{i=1}^{n} b_i \sigma_i = 1 \tag{2.5}$$

[$(a_1, \ldots, a_n)$ being pairwise relatively prime and a_i, b_i relatively prime for every i] is called a *Seifert fibered homology sphere* (in short, homology sphere or Sfh-sphere).

It is possible to calculate from formulas (2.2) and (2.4) a fundamental group of the orbifold:

$$\begin{aligned} \pi_1(S^2(\underline{a})) &= \pi(\Sigma(\underline{a})/S^1) = \pi(\Sigma(\underline{a}))/\langle h \rangle \\ &= \langle x_1, \ldots, x_n \,|\, x_i^{a_i} = 1, x_1 \cdots x_n = 1 \rangle \end{aligned} \tag{2.6}$$

where $\langle h \rangle$ is the center of $\pi(\Sigma(\underline{a}))$. This presentation shows that the group $\pi_1(S^2(\underline{a}))$ is isomorphic to a Fuchsian group of genus 0.

We give now the definition of a cobordism (see, for example, Freed and Uhlenbeck, 1984): Let M be a compact manifold. Submanifolds $N_0, N_1 \subseteq M$ are cobordant in M if such a compact submanifold $C \subseteq M \times [0, 1]$ exists that $\partial C = N_0 \times 0 \sqcup N_1 \times 1$.

The cobordism is both the equivalence relation between N_0 and N_1 and the space C itself. The factor [0, 1] may be interpreted as the time when the cobordism C transforms the submanifold N_0 to N_1.

3. ELEMENTARY COBORDISMS $W(\underline{a})$ and $W_0(\underline{a})$

Now we pass to the construction of cobordisms $W(\underline{a})$ and $W_0(\underline{a})$. They will be used as the main constructive blocks for creation of the 4-dimensional spaces describing topology changes in the ensemble of Sfh-spheres and the lens spaces which accompany the Sfh-spheres inevitably if the cobordisms do not have singular points. Factorization of the Sfh-sphere $\Sigma(\underline{a})$ by S^1 may be considered as an *orbit* map

$$w: \quad \Sigma(\underline{a}) \rightarrow S^2(\underline{a}) \tag{3.1}$$

[this results in the *orbifold* $S^2(\underline{a})$]. Let $\Sigma(\underline{a}) \times [0, 1]$ be a cylinder. Factorization of its "lower bound" $\Sigma(\underline{a}) \times 0$ by the pseudofree S^1-action gives a 4-dimensional space $W(\underline{a}) = W(a_1, \ldots, a_n)$ which is called a cylinder of the orbit map w. Since the S^1-action in $\Sigma(\underline{a})$ is pseudofree with the exception of n isolated exceptional orbits with isotropies $Z_{a_1}, \ldots, Z_{a_n}$, then $W(\underline{a})$ is a pseudofree orbifold (Fintushel and Stern, 1985) or V-manifold (Kawasaki, 1978). The orbifold $W(\underline{a})$ is a smooth manifold with the exception of n isolated singularities c_i whose neighborhoods are the cones $c_i \star L(a_i, b_i)$ on lens spaces $L(a_i, b_i)$, $i = 1, \ldots, n$, corresponding to the exceptional orbits in $\Sigma(a_1, \ldots, a_n) = \Sigma(\underline{a})$, which is the boundary of $W(\underline{a})$. Let $W_0(\underline{a})$ denote $W(\underline{a})$ with open cones around the singularities removed:

$$W_0(\underline{a}) = W(\underline{a}) - \bigsqcup_{i=1}^{n} \operatorname{int}(c_i \star L(a_i, b_i)) \tag{3.2}$$

We now have the smooth manifold $W_0(\underline{a})$ with a boundary

$$\partial W_0(\underline{a}) = -\bigsqcup_{i=1}^{n} L(a_i, b_i) \sqcup \Sigma(\underline{a}) \tag{3.3}$$

where the minus sign marks opposite orientations of the lens spaces. It is important to observe (Fintushel and Stern, 1990) that the fundamental group of $W_0(\underline{a})$ is isomorphic to $\pi_1(S^2(\underline{a}))$:

$$\pi_1(W_0(\underline{a})) = \langle x_1, \ldots, x_n \,|\, x_i^{a_i} = 1, x_1 \cdots x_n = 1 \rangle \tag{3.4}$$

The boundary $\partial W(\underline{a}) = \Sigma(\underline{a})$ can be interpreted as a spatial section of the universe created at the singular points of n cones on the lens spaces

$$\bigsqcup_{i=1}^{n} (c_i \star L(a_i, b_i))$$

with different vertices $\{c_i\}$. But it is possible to identify all these vertices $c_i \equiv c$:

$$c \star \left(\bigsqcup_{i=1}^{n} L(a_i, b_i) \right)$$

Thus from our point of view, the orbifold $W(\underline{a})$ is a cobordism describing a topology transformation of the type

$$\text{(vacuum)} \rightarrow \text{(Sfh-sphere)} \tag{3.5}$$

The cobordism $W_0(\underline{a})$ can be treated as a topology change from the lens spaces $\sqcup_{i=1}^{n} L(a_i, b_i)$ (in-state) to the universe with space section homeomorphic to the Sfh-sphere $\Sigma(a_1, \ldots, a_n) = \Sigma(\underline{a})$ (out-state) and conversely.

Gluing together different elementary cobordisms of types $W(\underline{a})$ and $W_0(\underline{a})$ will give us various examples of 4-orbifolds and 4-manifolds which exhibit topology transformations. For a correct definition of sewing operations we take advantage of one well-known topology operation known as *splicing*.

4. HOMOLOGY SPHERE SPLICING AND TOPOLOGY CHANGE SEWING

Let

$$\Sigma(\underline{a}_{0,i}) = \Sigma(a_0, a_1, \ldots, a_i)$$

and

$$\Sigma(\underline{a}_{i+1,n+1}) = \Sigma(a_{i+1}, \ldots, a_n, a_{n+1})$$

be two Sfh-spheres, where $a_1, \ldots, a_n$ are pairwise relative primes. Let TS_0 be an open tubular neighborhood of exceptional orbit S_0 in the Sfh-sphere $\Sigma(\underline{a}_{0,i})$, and TS_{n+1} be an open tubular neighborhood of exceptional orbit S_{n+1} in $\Sigma(\underline{a}_{i+1,n+1})$. We introduce the standard pairs of meridians and longitudes (m_0, l_0) and (m_{n+1}, l_{n+1}) on boundaries of

$$K_{0,i} = \Sigma(\underline{a}_{0,i}) - TS_0$$

and

$$K_{i+1,n+1} = \Sigma(\underline{a}_{i+1,n+1}) - TS_{n+1} \tag{4.1}$$

respectively. By sewing these manifolds according to rules $m_0 = l_{n+1}$ and $l_0 = m_{n+1}$, we obtain an Sfh-sphere

$$\Sigma(a_1, \ldots, a_n) = \Sigma(a_0, a_1, \ldots, a_i) \# \Sigma(a_{i+1}, \ldots, a_n, a_{n+1}) \tag{4.2}$$

if and only if

$$a_0 = a_{i+1} \cdots a_n; \qquad a_{n+1} = a_1 \cdots a_i \tag{4.3}$$

The operation (4.2) is known as splicing (Eisenbud and Neumann, 1985). A simple observation is that (4.3) is a condition for gluing Seifert structures on the boundaries of $K_{0,i}$ and $K_{i+1,n+1}$. The standard fiber on $\partial K_{0,i}$ is represented by

$$h_{0,i} = a_0 l_0 + \sigma_0 m_0$$

where $\sigma_0 = a_1 \cdots a_i$, and analogously the standard fiber on $\partial K_{i+1,n+1}$ is

$$h_{i+1,n+1} = a_{n+1} l_{n+1} + \sigma_{n+1} m_{n+1}$$

where $\sigma_{n+1} = a_{i+1} \cdots a_n$. Thus for sewing $h_{0,i} = h = h_{i+1,n+1}$, the equalities

$$a_0 = \sigma_{n+1} \qquad \text{and} \qquad a_{n+1} = \sigma_0 \tag{4.4}$$

[which are the same as in (4.3)] are sufficient.

If the splicing operation is repeated $(n - 2)$ times, one obtains an arbitrary Sfh-sphere from *simplest* Sfh-spheres, i.e., Sfh-spheres which have the minimal number $(m = 3)$ of exceptional orbits:

$$\begin{aligned} \Sigma(a_1, \ldots, a_n) &= \#_{i=1}^{n-2}\, \Sigma(a_1 \cdots a_i, a_{i+1}, a_{i+2} \cdots a_n) \\ &= \#_{i=1}^{n-2} \Sigma(a_1^i, a_{i+1}, a_{i+2}^n) \end{aligned} \tag{4.5}$$

(Here and below we use the abbreviations $a_1^i = a_1 \cdots a_i$; $a_i^n = a_i \cdots a_n$.) Thus, when considering topology changes, we naturally pay special attention to the following fragment of the splicing sum (4.5):

$$\Sigma(a_1^{i-1}, a_i, a_{i+1}, a_{i+2}^n) = \Sigma(a_1^{i-1}, a_i, a_{i+1}^n) \# \Sigma(a_1^i, a_{i+1}, a_{i+2}^n) \tag{4.6}$$

which we shall utilize for constructing a simple *sewed* cobordism $\hat{W}_{i,i+1}^s = \hat{W}(a_1^{i-1}, a_i, a_{i+1}, a_{i+2}^n)$, which exhibits the decomposition of a universe homeomorphic to one Sfh-sphere into two universes homeomorphic to a sum of two simplest Sfh-spheres.

Let us begin with a construction of orbifolds W_i, $W_{i,i+1}$, which are cylinders of the orbit maps:

$$w_i\colon\ \Sigma_i \to S_i^2; \qquad w_{i,i+1}\colon\ \Sigma_{i,i+1} \to S_{i,i+1}^2 \tag{4.7}$$

where

$$\begin{aligned} \Sigma_i &= \Sigma(a_1^{i-1}, a_i, a_{i+1}^n); & \Sigma_{i,i+1} &= \Sigma(a_1^{i-1}, a_i, a_{i+1}, a_{i+2}^n) \\ S_i^2 &= S^2(a_1^{i-1}, a_i, a_{i+1}^n); & S_{i,i+1}^2 &= S^2(a_1^{i-1}, a_i, a_{i+1}, a_{i+2}^n) \end{aligned} \tag{4.8}$$

By analogy with (3.2), we remove from the orbifolds W_i, $W_{i,i+1}$ the open cones around the singular points, and obtain elementary cobordisms:

$$W_{0i} = W_0(a_1^{i-1}, a_i, a_{i+1}^n); \qquad W_{0i,i+1} = W_0(a_1^{i-1}, a_i, a_{i+1}, a_{i+2}^n) \tag{4.9}$$

These cobordisms correspond to topology changes:

$$\text{(set of lens space)} \to \text{(Sfh-sphere)} \tag{4.10}$$

since they have the boundaries

$$\partial W_{0i} = -(L_1^{i-1} \sqcup L_i \sqcup L_{i+1}^n) \sqcup \Sigma_i \tag{4.11}$$

$$\partial W_{0i,i+1} = -(L_1^{i-1} \sqcup L_i \sqcup L_{i+1} \sqcup L_{i+2}^n) \sqcup \Sigma_{i,i+1} \tag{4.12}$$

where $L_1^i = L(a_1^i, b_1^i)$, $L_i = L(a_i, b_i)$, $L_{i+1}^n = L(a_{i+1}^n, b_{i+1}^n)$.

Changing the orientation of the cobordism $W_{0i,i+1}$ and sewing it together with cobordisms W_{0i} and W_{0i+1}, we obtain a new cobordism

$$W_{0i,i+1}^s = W_0^s(a_1^{i-1}, a_i, a_{i+1}, a_{i+2}^n) \tag{4.13}$$

The index s indicates that this cobordism has been *sewed* along the lens spaces L_1^{i-1}, L_i, L_{i+1}, L_{i+2}^n, whose invariants are written in parentheses. This cobordism has the boundary

$$\partial W_{0i,i+1}^s = -(\Sigma_{i,i+1} \sqcup L_1^i \sqcup L_{i+1}^n) \sqcup (\Sigma_i \sqcup \Sigma_{i+1}) \tag{4.14}$$

and can be put in correspondence with a topology change of type

$$-(\Sigma_{i,i+1} \sqcup L_1^i \sqcup L_{i+1}^n) = \Sigma^{\text{in}} \to \Sigma^{\text{out}} = (\Sigma_i \sqcup \Sigma_{i+1}) \tag{4.15}$$

A similar cobordism was first investigated by Siebenmann (1979), therefore the index s may be associated with the first letter of his surname.

It is possible "to remove" the lens spaces L_1^i and L_{i+1}^n off the cobordism boundary (4.14) if one glues along these lens spaces the cones

$$c_1 \star L_1^i \quad \text{and} \quad c_2 \star L_{i+1}^n, \quad \text{or} \quad c \star (L_1^i \sqcup L_{i+1}^n), \quad c_1 \equiv c \equiv c_2$$

We have obtained a cobordism $\hat{W}_{i,i+1}^s$ with two singular conic points c_1 and c_2 (one may identify $c_1 \equiv c \equiv c_2$). It "describes" a topology transformation

$$-\Sigma_{i,i+1} \sqcup c = \hat{\Sigma}^{\text{in}} \to \Sigma^{\text{out}} = \Sigma_i \sqcup \Sigma_{i+1} \tag{4.16}$$

where the caret over Σ^{in} and $\sqcup\, c$ tell us that the "in-state" has the singular point c. We can interpret the cobordism $\hat{W}_{i,i+1}^s$ in physical terms as splitting the universe $\Sigma_{i,i+1}$ (Sfh-sphere) into the universes Σ_i and Σ_{i+1} (two Sfh-spheres) induced by the vacuum creation of two lens spaces L_1^i and L_{i+1}^n.

We associate the topology change process

$$\Sigma(\underline{a}_{1,n}) \sqcup c = \hat{\Sigma}^{\text{in}} \to \Sigma^{\text{out}} = \Sigma(\underline{a}_{0,i}) \sqcup \Sigma(\underline{a}_{i+1,n+1}) \tag{4.17}$$

which corresponds to the splicing operation (4.2), with the cobordism $\hat{W}_0^s(a_0, a_1, \ldots, a_n, a_{n+1})$ with a boundary:

$$\partial \hat{W}_0^s(a_0, a_1, \ldots, a_n, a_{n+1}) = -\Sigma(\underline{a}_{1,n}) \sqcup \Sigma(\underline{a}_{0,i}) \sqcup \Sigma(\underline{a}_{i+1,n+1}) \tag{4.18}$$

This cobordism is a result of sewing together elementary cobordisms $W_0(a_1, \ldots, a_n)$, $W_0(a_0, a_1, \ldots, a_i)$, and $W_0(a_{i+1}, \ldots, a_n, a_{n+1})$ along the lens spaces $L(a_i, b_i)$, $i = 1, \ldots, n$, and pasting up two components $L(a_0, b_0)$, $L(a_{n+1}, b_{n+1})$ of its boundary by means of the cone $c \star (L(a_0, b_0) \sqcup L(a_{n+1}, b_{n+1}))$.

A more complicated case of splicing (4.5) is associated with a topology change

$$\Sigma(\underline{a}_{1,n}) \sqcup c = \hat{\Sigma}^{\text{in}} \to \Sigma^{\text{out}} = \bigsqcup_{i=1}^{n-2} \Sigma(a_1^i, a_{i+1}, a_{i+2}^n) \tag{4.19}$$

A cobordism describing this topology transformation is sewed of the *simplest* cobordisms $W_0(a_1^i, a_{i+1}, a_{i+2}^n)$, $i = 1, \ldots, n-2$, and of the cone on lens spaces

$$c \star \bigsqcup_{i=1}^{n-2} (L(a_1^i, b_1^i) \sqcup L(a_{i+1}^n, b_{i+1}^n))$$

5. SEWING TOGETHER EUCLIDEAN AND LORENTZIAN REGIONS

In Section 4 we have explicitly constructed cobordisms with Euclidean signature which can be interpreted in terms of topology changes. In other words, we actually consider topology transformations as a quantum tunneling phenomenon, i.e., transitions through classically forbidden regions, for example, $\hat{W}_0^s(a_0, a_1, \ldots, a_n, a_{n+1})$ with Euclidean signature. The picture of our semiclassical approach would be incomplete if we did not define the procedure of sewing together Euclidean and Lorentzian regions along a boundary, which in our case is a disjoint union of Sfh-spheres and, sometimes, lens spaces. Let us discuss sufficient boundary conditions for sewing gravitational fields, expressed via *Ashtekar variables* in connection representation.

The classical canonical variables of Ashtekar *et al.* (1989) (see also Jacobson and Smolin, 1988) are an $SO(3, C)$ spatial connection A_a^i and a set of triads (or frame fields) of a foliation of space-time $\tilde{E}_i^a$, where a is a spatial index on a spacelike section Σ and i is a flat Euclidean index which can be thought of as an $SO(3, C)$ index (a tilde denotes the density weight). The $SO(3, C)$-connection may be identified with a spatial pullback of the self-dual part of the spin-connection.

These variables parametrize the phase space of complex general relativity. A real metric with Lorentzian signature may be recovered by imposing appropriate reality conditions (Ashtekar *et al.*, 1989): $\tilde{E}_i^a \tilde{E}^{bi}$ is real, and its time derivative is real, too. A metric of Euclidean signature is obtained by taking A_a^i and $\tilde{E}_i^a$ as real (Capovilla *et al.*, 1995). Thus in the Euclidean regime we can obtain a $SO(3)$-bundle V with the connection A_a^i over the 4-dimensional manifold $\hat{W}_0$. This $SO(3)$-bundle is lifted to an $SU(2)$-bundle if the second Stiefel–Whitney class $w_2(V)$ vanishes. In this case the gravitational instantons are described exactly as $SU(2)$ gauge fields (Capovilla *et al.*, 1990).

The observations stated above along with the results of Halliwell and Hartle (1990) and Fujiwara *et al.* (1992) (on vanishing of an extrinsic curvature $K_{ab} = 0$ on a boundary between Euclidean and Lorentzian signature regions) yield the following conclusion: *the sufficient conditions for sewing together Euclidean and Lorentzian domains consist of triviality of the connection A_a^i and reality of $\tilde{E}_i^a$ on each component of the boundary.* We have restricted our consideration (or approximation) to the equivalence classes of flat connections over Euclidean cobordisms (of $\hat{W}^s$ type). To obtain a *trivial* connection Θ on the boundary

$$\partial \hat{W}^s = \Sigma^{\text{in}} \sqcup \Sigma^{\text{out}} \tag{5.1}$$

it is sufficient to sew collars, i.e., to construct the cobordism (see, for example, Freed and Uhlenbeck, 1984)

$$\hat{W}^{s,\text{coll}} = (\Sigma^{\text{in}} \times [-1, 0]) \bigcup_{\Sigma^{\text{in}}} \hat{W}^s \bigcup_{\Sigma^{\text{out}}} (\Sigma^{\text{out}} \times [0, 1]) \tag{5.2}$$

and to define the paths

$$A_t = (1 + t)A^{\text{in}} - t\Theta, \qquad t \in [-1, 0] \tag{5.3}$$

$$A_t = (1 - t)A^{\text{out}} + t\Theta, \qquad t \in [0, 1] \tag{5.4}$$

of connections on the collars (Okonek, 1991).

Furthermore, to the boundaries of the cobordism $\hat{W}^{s,\text{coll}}$, the "ends"

$$\Sigma^{\text{in}} \times \mathbf{R}_- \qquad \text{and} \qquad \Sigma^{\text{out}} \times \mathbf{R}_+$$

with the Lorentzian signature (where $\mathbf{R}_- = (-\infty, -1]$; $\mathbf{R}_+ = [+1, +\infty)$) are pasted up. At the points $t = -1 \in \mathbf{R}_-$ and $t = +1 \in \mathbf{R}_+$, the Lorentzian connection is *trivial.*

If the lens spaces $L(a_i, b_i)$ enter the boundary of a cobordism, then they must be pasted up by collars and thereupon by Lorentzian "ends."

6. FLAT CONNECTIONS OVER COBORDISMS

6.1. Wave Functions in the Connection and Loop Representations

There exists a well-known solution to all the constraint equations of general relativity with a cosmological constant in the connection representation (Kodama, 1990; Brügmann *et al.*, 1992). This is the exponent of the Chern–Simons functional

$$\Psi[A] = \exp\left(-\frac{1}{\lambda} CS[A]\right) \tag{6.1}$$

$$CS[A] = \frac{1}{8\pi^2} \int \epsilon^{abc} \operatorname{Tr}\left(A_a \partial_b A_c + \frac{2}{3} A_a A_b A_c\right) \tag{6.2}$$

Serious arguments were given (Smolin, 1995) in support of the proposal that all physical quantum states in the loop representation are expressed by means of the Kodama states via the transform

$$\Psi[\gamma] = \int [dA] \exp\left(-\frac{1}{\lambda} CS[A]\right) \operatorname{Tr} \mathrm{P} \exp \oint_\gamma A_a\, dx^a \tag{6.3}$$

where γ is a loop in a spacelike section Σ.

We restrict ourselves to calculation of the path integral (6.3) in the stationary phase approximation. The relation

$$\frac{\delta}{\delta A_a^i} \Psi[A] = \frac{1}{4\pi^2 \lambda} \epsilon^{abc} F^i_{bc} \Psi[A] \tag{6.4}$$

demonstrates that the flat connections ($F^i_{ab} = 0$) are critical points of the Chern–Simons functional [as a Morse function on the orbit space of connections over Σ modulo gauge invariance (Okonek, 1991)]. Consequently in the connection representation the flat connections are stationary points of the wave function phase. Thus we restrict ourselves to evaluation of the path integrals on a modulo space $R(\Sigma)$ of flat $SU(2)$-connections over a Seifert fibered 3-manifold Σ(Fintushel and Stern, 1990).

We recall several key observations about the modulo space $R(\Sigma)$ of flat $SU(2)$-connections over Σ (Kirk and Klassen, 1990; Okonek, 1991; Saveliev, 1992).

1. The space of flat $SU(2)$-connections (modulo gauge equivalence) is homeomorphic to the space of representations of the fundamental group of Σ in $SU(2)$ (modulo conjugation):

$$R(\Sigma) \cong \{\alpha \,|\, \alpha\colon \pi_1(\Sigma) \to SU(2)\}/\mathrm{conj} \tag{6.5}$$

2. Two flat connections A_1 and A_2 which lie on the same component

of the space $R(\Sigma)$ have the same Chern–Simons invariant $CS[A_1] \equiv CS[A_2]$ (mod 4).

3. Let Σ be a Sfh-sphere $\Sigma(\underline{a}) = \Sigma(a_1, \ldots, a_n)$; then the set of connection components of $R(\Sigma(\underline{a}))$ is in one-to-one correspondence with the set of admissible collections of the rotation numbers $(\underline{l}) = (l_1, \ldots, l_n)$, which completely specify the class α of irreducible representations of $\pi_1(\Sigma(\underline{a}))$ in $SU(2)$. Thus $CS[A] = CS(\underline{l})$.

Furthermore, for a flat connection, the holonomy along a loop depends only on the homotopy class of the loop and consequently is defined on the elements of $R(\Sigma(\underline{a}))$. This means that loop variables of Rovelli and Smolin (1990)

$$\mathrm{Tr}\, \alpha(\gamma) = \mathrm{Tr}\, \mathrm{P} \exp \oint_\gamma A_a\, dx^a \tag{6.6}$$

are defined uniquely on the loop homotopy classes. A basis in the loop variable space (i.e., in a configuration space of Wilson loops) consists of the homotopies along nontrivial loops x_i which generate the fundamental group $\pi_1(\Sigma(\underline{a}))$ in the form (2.4). A complete collection of invariants that specify the representation α is $\{\mathrm{Tr}\, \alpha(x_i) \mid i = 1, \ldots, n\}$.

4. Fintushel and Stern (1990) have demonstrated that these invariants are determined by the rotation numbers $(\underline{l}) = (l_1, \ldots, l_n)$:

$$\mathrm{Tr}\, \alpha(x_i) = 2 \cos(\pi l_i / a_i) \tag{6.7}$$

and consequently are fixed in the limits of the same component of the space $R(\Sigma(\underline{a}))$.

In this case the value $\mathrm{Tr}\, \alpha(\gamma) = \mathrm{Tr}\, \Pi_i\, \alpha(x_i)$ is determined also uniquely by a connected component of $R(\Sigma(\underline{a}))$ [or, equivalently, by $(\underline{l})$].

Finally, each rotation number l_i is an element of Z_{a_i}, hence it has a finite number of different values. Thus the path integral (6.3) in the stationary phase approximation is represented as a finite sum of contributions of connected components of the flat connection moduli space $R(\Sigma(\underline{a}))$ (Rozansky, 1995), i.e., the integral (6.3) is reduced to a sum over admissible collections $(\underline{l}) = (l_1, \ldots, l_n)$:

$$\Psi[\gamma] = \sum_{(\underline{l})} C(\underline{l}) \exp\left(-\frac{1}{\lambda} CS(\underline{l}) \right) \mathrm{Tr}\, \alpha(\gamma) \tag{6.8}$$

where $C(\underline{l})$ is a weight multiplier. When $\gamma = x_i$ we have the explicit expression

$$\Psi[x_i] = \sum_{(\underline{l})} C(\underline{l}) \exp\left(-\frac{1}{\lambda} \frac{e^2}{a} \right) 2 \cos\left(\frac{\pi l_i}{a_i} \right) \tag{6.9}$$

where

$$e = \sum_{i=1}^{n} l_i \sigma_i; \qquad a = a_1 \cdots a_n; \quad \sigma = a/a_i \tag{6.10}$$

Now let us consider a cobordism $\hat{W}_0^s$ with a boundary

$$\partial \hat{W}_0^s = \Sigma_1 \sqcup \cdots \sqcup \Sigma_M \sqcup (-\Sigma_{M+1}) \sqcup \cdots \sqcup (-\Sigma_N) = \Sigma^{\text{out}} \sqcup \Sigma^{\text{in}} \tag{6.11}$$

where

$$\Sigma^{\text{out}} = \bigsqcup_{k=1}^{M} \Sigma_k$$

and

$$\Sigma^{\text{in}} = -\bigsqcup_{k=M+1}^{N} \Sigma_k$$

Let ϵ_k be given by

$$\begin{aligned} \epsilon_k &= +1 \qquad \text{if} \quad k = 1, \ldots, M \\ \epsilon_k &= -1 \qquad \text{if} \quad k = M+1, \ldots, N \end{aligned} \tag{6.12}$$

In this case the Chern–Simons invariant of the cobordism $\hat{W}_0^s$ is of the form

$$CS[A] = \frac{1}{8\pi^2} \int_{\hat{W}_0^s} F_A \wedge F_A$$

$$\frac{1}{8\pi^2} \int_{\partial \hat{W}_0^s} \epsilon^{abc} \operatorname{Tr}\left(A_a \partial_b A_c + \frac{2}{3} A_a A_b A_c \right) = \sum_{k=1}^{N} \epsilon_k CS(\underline{l}^k) \tag{6.13}$$

In this expression $(\underline{l}^k) = (l_1^k, \ldots, l_n^k)$ is a collection of admissible rotation numbers for boundary component with the number k.

The Kodama wave function in the connection representation (6.1) for the Sfh-sphere $\Sigma(\underline{a})$ is

$$\Psi(\underline{l}) = C(\underline{l}) \exp\left(-\frac{1}{\lambda} CS(\underline{l}) \right) \tag{6.14}$$

and it may be generalized in the case of the cobordism $\hat{W}_0^s$ with the multicomponent boundary (6.11) (see, for example, Section 6 of Dijkgraaf and Witten, 1990) as

$$\Psi(\underline{l}^1, \ldots, \underline{l}^N) = \prod_{k=1}^{N} C(\underline{l}^k) \exp\left(-\frac{\epsilon_k}{\lambda} CS(\underline{l}^k) \right) \tag{6.15}$$

It is possible to choose the orientation of the cobordism $\hat{W}_0^s$ so that

$$\sum_{k=1}^{N} \epsilon_k CS(\underline{l}^k) \geq 0 \tag{6.16}$$

The wave function (*à la* Kodama) (6.15) in this case is a topology changing amplitude corresponding to the cobordism $\hat{W}_0^s$.

6.2. Homology Sphere from Vacuum ("Nothing")

Let us consider (without details) the simplest topology change (3.5)

$$(\text{vacuum}) \equiv c \to \Sigma(a_1, \ldots, a_n) \equiv (\text{Sfh-sphere}) \tag{6.17}$$

We recall that the cobordism $W(\underline{a}) = W(a_1, \ldots, a_n)$ which describes this topology transformation has n singular points with neighborhoods homeomorphic to the cones on lens spaces $L(a_i, b_i)$ ($i = 1, \ldots, n$). The topology change amplitude of the process (6.17) is

$$\Psi(\underline{l}) = C(\underline{l}) \exp\left(-\frac{1}{\lambda} CS(\underline{l})\right) \tag{6.18}$$

where $\underline{l}$ is an admissible collection of rotation numbers $(l_1, \ldots, l_n)$. One would like to interpret the wave function (6.18) as a probability amplitude of creation of a universe with Sfh-sphere spatial section $\Sigma(a_1, \ldots, a_n)$ from the singular vacuum state c through the cone $\sqcup_{i=1}^n (c \star L(a_i, b_i))$. In this case it is necessary to be cautious. As is well known, the Chern–Simons functional $CS[A]$ is not invariant under large gauge transformations. Then, it transforms as

$$CS[A] \to CS[A] + 4n, \qquad n \in \mathbf{Z} \tag{6.19}$$

By virtue of this indefiniteness, the exponent in the wave function (6.18) is also indefinite. We suggest to use a "cosmological constant" λ to compensate this indefiniteness. In the normalization of the $CS[A]$ accepted here (Fintushel and Stern, 1990), the gauge-invariant value of the Chern–Simons functional is defined modulo 4. Fintushel and Stern (1990) and Kirk and Klassen (1990) have demonstrated that

$$CS[A] = CS(\underline{l}) \equiv e^2/a \pmod 4, \qquad \text{where} \quad e = \sum_{i=1}^{n} l_i \sigma_i \tag{6.20}$$

We determine the Chern–Simons invariant value in the exponent (6.18) as *the least nonnegative residue* of $CS(\underline{l})$ modulo 4. In this case it is natural to define the magnitude of the "cosmological constant" as a preferred value of

Chern–Simons invariant for the Sfh-sphere $\Sigma(\underline{a})$. In their article, Fintushel and Stern (1990) calculated the quantity

$$\tau(\Sigma(\underline{a})) = \min\{CS[A] \,|\, A \in R(\Sigma(\underline{a}))\} = \frac{1}{a_1 \cdots a_n} = \frac{1}{a} \tag{6.21}$$

We think that λ may be defined as

$$\lambda = \tau(\Sigma(\underline{a})) = \frac{1}{a} \tag{6.22}$$

Thus the expression (6.18) takes the form

$$\Psi(\underline{l}) = C(\underline{l}) \exp(-e^2) \tag{6.23}$$

where e^2 is in fact the *least nonnegative residue* of $CS(\underline{l})/\lambda$ (mod $4a$).

It is easy to demonstrate (Fintushel and Stern, 1990; Kirk and Klassen, 1990) that

$$e^2 \equiv \sum_{i=1}^{n} (l_i\sigma_i)^2 \pmod{4a} \tag{6.24}$$

Thus in the stationary phase approximation a partial factorization of the universe creation amplitude takes place:

$$\Psi(\underline{l}) = C(\underline{l}) \prod_{i=1}^{n} \exp[-(l_i\sigma_i)^2] \tag{6.25}$$

Kirk and Klassen (1990) also calculated the Chern–Simons invariants of lens spaces. Their results give the possibility to rewrite $\Psi(\underline{l})$ as

$$\Psi(\underline{l}) = C(\underline{l}) \prod_{i=1}^{n} \exp\left(-\frac{1}{\lambda}\, CS(L(a_i, b_i))\right) \tag{6.26}$$

Birmingham (1995) has observed such a factorization property of the topology changing amplitude in a similar topology situation, but in another approximation [the Regge calculus approach in a simplicial minisuperspace for the cone $c \star (\sqcup_{i=1}^{K} L(p_i, 1))$].

In order to change from a Euclidean regime to a Lorentzian one, it is necessary to glue up a collar $\Sigma(\underline{a}) \times [0, 1]$ with connection $A_t = (1 - t)A(\underline{l}) + t\Theta$, where $A(\underline{l})$ is a flat connection over the boundary $\partial W(\underline{a})$ (to which the collar is glued), and Θ is a trivial connection. (The general method has been described in Section 5.) Thus it is natural to suggest that the transition amplitude for the process $c \to \Sigma(\underline{a})_{\text{Lor}}$ is equal to

$$\Psi(\Sigma(\underline{a})_{\text{Lor}}) = \sum_{\underline{l}} C(\underline{l}) \exp(-e^2) \tag{6.27}$$

where $\Sigma(\underline{a})_{\mathrm{Lor}}$ is a spacelike section in the Lorentzian "end" $(\Sigma(\underline{a}) \times \mathrm{R}_+)_{\mathrm{Lor}}$ pasted up to a boundary $\Sigma(\underline{a}) \times \{1\}$ of the collar. In (6.27), the summation is taken over admissible collections $(\underline{l}) = (l_1, \ldots, l_n)$ of rotation numbers.

It may be also suggested that the weight coefficients $C(\underline{l})$ are proportional to C^{2m-6}, where C = const, and m is the number of the rotation numbers in the admissible collection $(\underline{l}) = (l_1, \ldots, l_n)$ which are not zero. This estimate is made in accordance with the fact that the connected component of the moduli space [with the collection $(l_1, \ldots, l_n)$] has dimension $2m - 6$ (Fintushel and Stern, 1990).

6.3. Splitting of a Universe: A General Consideration and Examples

The situation is more complicated when a topology change cobordism $\hat{W}_0^s$ is sewed by several elementary cobordisms of type $W_0(\underline{a})$ (see Section 4). In order to calculate the topology changing amplitude (6.15) it is necessary to define the procedure of gluing up connections in $SO(3)$-bundles which are prescribed over elementary cobordisms, since not all of the rotation numbers $\underline{l}_1, \ldots, \underline{l}_N$ are independent. In order to sew the flat connections over cobordisms (orbifolds) we shall use the methods of Fintushel and Stern (1987, 1990) and Saveliev (1992).

If $\Sigma(\underline{a})$ is a Seifert fibered homology sphere, then there exists a one-to-one correspondence between equivalence classes of representations α of $\pi_1(\Sigma(\underline{a}))$ into $SU(2)$ and equivalence classes of representations $\tilde{\alpha}$ of $\pi_1(W_0(\underline{a}))$ into $SO(3)$, which make up the moduli space $R(W_0(\underline{a}))$. Thus $R(\Sigma(\underline{a})) \cong R(W_0(\underline{a}))$ (Fintushel and Stern, 1990; Okonek, 1991). Then each $\tilde{\alpha} \in R(W_0(\underline{a}))$ assigns $SO(3)$-bundle V_α over elementary cobordism $W_0(\underline{a})$ with flat connection $A_{\tilde{\alpha}}$, which is fixed by an admissible collection of rotation numbers $(\underline{l}) = (l_1, \ldots, l_n)$, $l_i \in \mathrm{Z}_{a_i}$. In the simplest case of the homology spheres with three exceptional orbits $\Sigma(p, q, r)$, the space $R(W_0(p, q, r))$ has a finite number of points. Therefore in this case the flat connections are isolated exceptional points of the Chern–Simons functional such as the Morse function with nondegenerate critical points on the space of all connections over $W_0(p, q, r)$ (Fintushel and Stern, 1990). Calculation schemes for the admissible collections $(l_1, \ldots, l_n)$ are developed by Fintushel and Stern (1990) and Kirk and Klassen (1990). In particular, the admissible collection must satisfy the following conditions:

$$\begin{aligned} &l_i \text{ is even if } b_i \text{ is even, or } \alpha(h) = +1 \\ &l_i \text{ is odd if } b_i \text{ is odd, and } \alpha(h) = -1 \end{aligned} \tag{6.28}$$

where b_i are defined by $b_i\sigma_i \equiv 1 \pmod{a_i}$.

Over the lens space $L(a_i, b_i) \subset \partial W_0(\underline{a})$ an $SO(3)$-bundle V_α is restricted to a bundle $L_{\alpha,i} \oplus R$, where R is a trivial bundle, and $L_{\alpha,i}$ is a $U(1)$-bundle, which is characterized by the Euler class

$$e \in H^2(L(a_i, b_i)) \cong Z_{a_i} \tag{6.29}$$

To extend the $SO(3)$-bundle from one elementary cobordism W_{0A} to the other W_{0B}, the critical observation (Fintushel and Stern, 1990; Saveliev, 1992) is that the Euler numbers which classify the $U(1)$-bundle over $L(a_i, b_i)$ are expressed in terms of the same rotation numbers which describe the $SO(3)$-bundle over the elementary cobordism $W_0(\underline{a}) = W_0(a_1, \ldots, a_n)$. We choose a basis in $\pi_1(L(a_i, b_i)) \cong Z_{a_i}$ with generator

$$g_i = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{b_i} \end{pmatrix}, \qquad \zeta = \exp(2\pi i/a_i) \tag{6.30}$$

acting in the covering $S^3 \to L(a_i, b_i)$. In this basis the Euler number of the bundle $L_{\alpha,i}$ is

$$e \equiv l_i (\text{mod } a_i) \tag{6.31}$$

Both to the present end and for better understanding of the bundle structure over cobordism $W(\underline{a}) = W(a_1, \ldots, a_n)$ [corresponding to the topology change (6.17)] it is important to remember that the bundle $L_{\alpha,i}$ is extended through the boundary component $L(a_i, b_i) \subset \partial W_0(\underline{a})$ over the cone $c \star L(a_i, b_i)$ as a $SO(3)$-V-bundle (Fintushel and Stern, 1987). The rotation number of this bundle over the conic point $y_i \in W(\underline{a})$ is l_i with respect to the generator g_i, (6.30). I think that this is one of the reasons that the universe creation amplitude (6.18) factorizes to (6.26).

Let each cobordism of W_{0A} and W_{0B} have lens spaces $L(a_i, b_i)$, $i = 1, \ldots, k$, as components of its boundary, i.e.,

$$\partial W_{0A} \supset L(a_i, b_i) \subset \partial W_{0B}, \qquad i = 1, \ldots, k$$

We glue together this cobordisms along $\sqcup_{i=1}^{k} L(a_i, b_i)$. In order to glue the cobordisms with $SO(3)$-bundles over them, it is necessary and sufficient to satisfy the conditions

$$e^A \equiv e^B \quad (\text{mod } a_i), \qquad i = 1, \ldots, k \tag{6.32}$$

for the Euler numbers, or equivalently

$$l_i^A \equiv l_i^B \quad (\text{mod } a_i), \qquad i = 1, \ldots, k \tag{6.33}$$

for the rotation numbers.

For example, we consider now the cobordism (orbifold)

$$\hat{W}^s_{i,i+1} = \hat{W}(a_1^{i-1}, a_i, a_{i+1}, a_{i+2}^n)$$

(see Section 4), which is the result of sewing together the cobordisms $\hat{W}_{0i}$, $\hat{W}_{0i+1}$, and $W_{0i,i+1}$, where the caret indicates that the lens spaces have been glued up by the cone $c \star (L(a_1^i, b_1^i) \sqcup L(a_{i+1}^n, b_{i+1}^n))$. It is natural to separate the boundary

$$\partial \hat{W}^s_{0i,i+1} = (-\Sigma_{i,i+1}) \sqcup (\Sigma_i \sqcup \Sigma_{i+1}) \tag{6.34}$$

into

$$\Sigma^{\text{in}} = -\Sigma_{i,i+1} \qquad \text{and} \qquad \Sigma^{\text{out}} = \Sigma_i \sqcup \Sigma_{i+1} \tag{6.35}$$

Then we assume that an admissible collection $(\underline{l})^{\text{in}} = (l_1^{i-1}, l_i, l_{i+1}, l_{i+2}^n)$ of the rotation numbers is given on the in-hypersurface Σ^{in}, i.e., we fix a flat $SU(2)$-connection over Σ^{in}.

It is useful to recall that for the Sfh-sphere $\Sigma_{i,i+1}$ there is a one-to-one correspondence between the flat $SU(2)$-connection over $\Sigma_{i,i+1}$ and the flat $SO(3)$-connections over $W_{0i,i+1}$ (Fintushel and Stern, 1990; Okonek, 1991). Therefore the admissible collection of rotation numbers $(\underline{l})^{\text{in}}$ can be extended over $W_{0i,i+1}$. Thus we obtain a class of static solutions of Einstein's equations with a flat $SO(3)$-connection corresponding to $(\underline{l})^{\text{in}}$. [See an analogous situation in Section 5 of Smolin (1989).] This $SO(3)$-connection is restricted over $L(a_1^{i-1}, b_1^{i-1})$ and $L(a_i, b_i) \subset \partial W_{0i,i+1}$ to a $U(1)$-connection with rotation numbers $l_1^{i-1}(\text{mod } a_1^{i-1})$ and $l_i(\text{mod } a_i)$, respectively. The sewing condition for the connections over $W_{0i,i+1}$ and $\hat{W}_{0i}$ means that over $L(a_1^{i-1}, b_1^{i-1})$ and $L(a_i, b_i) \subset \partial \hat{W}_{0i}$, a $U(1)$-connection with similar rotation numbers l_1^{i-1} (mod a_1^{i-1}) and $l_i(\text{mod } a_i)$ must be defined. The rotation number l_{i+1}^n remains arbitrary up to admissibility of the collection $(l_1^{i-1}, l_i, l_{i+1}^n)$, which should define a flat $SO(3)$-connection over the cobordism $\hat{W}_{0i}$. Analogously the rotation numbers l_{i+1}, l_{i+2}^n are "translated" from the elementary cobordism $W_{0i,i+1}$ to the cobordism $\hat{W}_{0i+1}$ with sewing condition of type (6.33) over the lens spaces $L(a_{i+1}, b_{i+1})$ and $L(a_{i+2}^n, b_{i+2}^n)$. The rotation number l_1^i remains arbitrary up to that the collection $(l_1^i, l_{i+1}, l_{i+2}^n)$ must be admissible over the cobordism $\hat{W}_{0i+1}$.

The dimension of each connection component of the moduli space $R(\Sigma_i)$ is equal to zero $(2m - 6 = 0)$ when $m = 3$. Thus it is natural to put

$$C(l_1^{i-1}, l_i, l_{i+1}^n) = C(l_1^i, l_{i+1}, l_{i+2}^n) = C^{2m-6} = 1 \tag{6.36}$$

in formula (6.15) for the topology change amplitude of the process (4.16). Therefore the amplitude of this process is

$$\Psi(l_1^{i-1}, l_i, l_{i+1}, l_{i+2}^n, l_1^i, l_{i+1}^n) = \Psi(\underline{l}^k) = C_{i,i+1} \exp\left(-\frac{1}{\lambda} \sum_k \epsilon_k CS(\underline{l}^k)\right) \tag{6.37}$$

(the connections are fixed over Σ^{in} and Σ^{out}), where $C_{i,i+1} = C(l_1^{i-1}, l_i, l_{i+1}, l_{i+2}^n)$ and in this case

$$\sum_k \epsilon_k CS(\underline{l}^k) = CS(l_1^{i-1}, l_i, l_{i+1}^n) + CS(l_1^i, l_{i+1}, l_{i+2}^n) - CS(l_1^{i-1}, l_i, l_{i+1}, l_{i+2}^n) \qquad (6.38)$$

Further we utilize the general expression for the Sfh-sphere Chern–Simons invariant (Fintushel and Stern, 1990; Kirk and Klassen, 1990)

$$CS(\underline{l}^k) \equiv e^2/a \pmod 4, \qquad \text{where } e = \sum_{i=1}^n l_i\sigma_i \qquad (6.39)$$

Taking into account (6.28), we have in the case under consideration

$$\sum_k \epsilon_k CS(\underline{l}^k) \equiv [(l_1^i a_{i+1}^n)^2 + (l_{i+1}^n a_1^i)^2]/a \pmod 4 \qquad (6.40)$$

If we use our definition (6.22) of the "cosmology constant" for the cobordism $\hat{W}_{i,i+1}^s$, then

$$\frac{1}{\lambda}\sum_k \epsilon_k CS(\underline{l}^k) \equiv (l_1^i a_{i+1}^n)^2 + (l_{i+1}^n a_1^i)^2 \pmod{4a} \qquad (6.41)$$

Finally the formula (6.37) is factorized *completely* with respect to the boundary components of cobordism $\hat{W}_{i,i+1}^s$. Then we have

$$\Psi(\underline{l}^k) = C_{i,i+1}\exp[-(l_1^i a_{i+1}^n)^2 - (l_{i+1}^n a_1^i)^2] = C_{i,i+1}\exp\left[-\frac{1}{\lambda}CS(L(a_1^i, b_1^i)) - \frac{1}{\lambda}CS(L(a_{i+1}^n, b_{i+1}^n))\right] \qquad (6.42)$$

where $CS(L(a, b))$ is the Chern–Simons invariant of the lens space $L(a, b)$ (Kirk and Klassen, 1990).

According to the general method (see Section 5), we glue collars to Σ^{in} and Σ^{out}. Along these collars the connections A^{in} and A^{out} reduce to the trivial connection Θ in accordance with (5.3) and (5.4). Then it is possible to paste up the new boundaries by the "ends" $\Sigma^{in} \times R_-$ and $\Sigma^{out} \times R_+$ with the Lorentzian signature. In this case the topology change amplitude of the process

$$(\Sigma^{in})_{Lor} \to (\Sigma^{out})_{Lor} \qquad (6.43)$$

[through Euclidean instanton $\hat{W}_{i,i+1}^s$ with a flat $SO(3)$-connection in the stationary phase approximation] is

$$\Psi[\hat{W}_{i,i+1}^s] = \sum_{(l_1^{i-1}, l_i, l_{i+1}, l_{i+2}^n)} C(l_1^{i-1}, l_i, l_{i+1}, l_{i+2}^n) \sum_{(l_1^i, l_{i+1}^n)} \exp[-(l_1^i a_{i+1}^n)^2 - (l_{i+1}^n a_1^i)^2] \qquad (6.44)$$

where the summations are taken over the admissible collections of rotation numbers $(l_1^{i-1}, l_i, l_{i+1}, l_{i+2}^n)$, $(l_1^{i-1}, l_i, l_{i+1}^n)$ and $(l_1^i, l_{i+1}, l_{i+2}^n)$, which fix the flat connections over $\Sigma_{i,i+1}$, Σ_i, and Σ_{i+1}, respectively. Thus the flat connection contribution to the topology change amplitude of the process (4.16)

$$(\text{Sfh-sphere}) \rightarrow (\text{two Sfh-spheres})$$

is expressed, up to the constant

$$C_4 = \sum_{(l_1^{i-1}, l_i, l_{i+1}, l_{i+2}^n)} C(l_1^{i-1}, l_i, l_{i+1}, l_{i+2}^n) \tag{6.45}$$

by the amplitude of topology transformation

$$(\text{vacuum}) \rightarrow (\text{two lens spaces})$$

which is

$$\begin{aligned} \Psi[\hat{W}^s_{i,i+1}] &= \sum_{(l_1^i, l_{i+1}^n)} C_4 \exp[-(l_1^i a_{i+1}^n)^2 - (l_{i+1}^n a_1^n)^2] \\ &= \sum_{(l_1^i, l_{i+1}^n)} C_4 \exp\left[-\frac{1}{\lambda} CS(L(a_1^i, b_1^i)) - \frac{1}{\lambda} CS(L(a_{i+1}^n, b_{i+1}^n))\right] \end{aligned} \tag{6.46}$$

It is possible to generalize these amplitudes to the case of creation of a universe with the spatial section homeomorphic to the Sfh-sphere $\Sigma(\underline{a}) = \Sigma(a_1, \ldots, a_n)$ out of $(n - 2)$ simplest Sfh-spheres (4.5) with three exceptional orbits (Saveliev, 1992):

$$\Sigma^{\text{in}} = \bigsqcup_{i=1}^{n-2} \Sigma(a_1^i, a_{i+1}, a_{i+2}^n) \rightarrow \Sigma(a_1, \ldots, a_n) = \Sigma^{\text{out}} \tag{6.47}$$

In this situation the topology change amplitude also factorizes and is given by

$$\begin{aligned} \Psi[\hat{W}^s(a_1, \ldots, a_n)] &= \sum_{(l_1^i, l_{i+1}^n)} C_n \prod_{i=1}^{n-2} \exp[-(l_1^i a_{i+1}^n)^2 - (l_{i+1}^n a_1^n)^2] \\ &= \sum_{(l_1^i, l_{i+1}^n)} C_n \prod_{i=1}^{n-2} \exp\left[-\frac{1}{\lambda} CS(L(a_1^i, b_1^i)) - \frac{1}{\lambda} CS(L(a_{i+1}^n, b_{i+1}^n))\right] \end{aligned} \tag{6.48}$$

where

$$C_n = \sum_{(l_1, \ldots, l_n)} C(l_1, \ldots, l_n) \tag{6.49}$$

The summations in (6.48) are taken over all pairs (l_1^i, l_{i+1}^n) which are contained in the admissible collections $(l_1^i, l_{i+1}, l_{i+2}^n)$, $i = 1, \ldots, n - 2$. In (6.49) the summation is taken over all admissible collections $(\underline{l}) = (l_1, \ldots, l_n)$ (Fintushel and Stern, 1990; Saveliev, 1992). We recall that in the cobordism $\hat{W}^s(a_1, \ldots, a_n)$ the rotation numbers are "translated" through the lens spaces $L(a_i, b_i)$, $i = 1, \ldots, n$.

7. DISCUSSION AND CONCLUSION

Factorization of topology changing amplitudes has been obtained by Birmingham (1995) for a cone on the disjoint union of lens spaces within the Regge calculus approach in the frame of the simplicial minisuperspace approximation. Birmingham indicated that this factorization property is a direct consequence of the restrictive nature of Wheeler–DeWitt minisuperspace and the simplicity of the cone-type cobordism. We have complicated the topology of cobordism and changed the approximation. However, the factorization properties are conserved both in the case of topology change (6.17) (vacuum) → (Sfh-sphere) [thanks to (6.24)] and for the processes of splitting or fusion (creation) of a universe which were investigated in Section 6.3 [due to congruences of type (6.41)]. It is interesting that the Hartle–Hawking-type wave functions (6.26) or (6.27) describe the creation of a universe with its spatial section homeomorphic to the Sfh-sphere $\Sigma(a_1, \ldots, a_n)$, through the cone $c \star \sqcup_{i=1}^{n} L(a_i, b_i)$ on the lens spaces. The variety of the topology properties of this universe is determined first of all by the pairwise relative prime numbers $(a_1, \ldots, a_n)$. What physical quantities may correspond to functions of the collection $(a_1, \ldots, a_n)$? The author hopes that these quantities would be the coupling constants of the fundamental interactions. Some ideas on this point may be found in Efremov and Shchyotochkin (1986). Moreover, in the models studied in this article (see Section 6), the cosmological constant in Kodama's solution is assumed to be equal to the minimal value (6.21) of the Chern–Simons invariant of the topology change cobordism. This approach to the definition of the cosmological constant is based on the internal logic of our models, since it is necessary to avoid uncertainty in the topology changing amplitudes in the Euclidean regime. If one could justify this point more strictly, a new path would be probably open to evaluate the other fundamental coupling constants [first of all, the gravitational constant (Weinberg, 1989)] by means of the topological invariants of cobordisms describing topology changes.

Since the ensemble of homology spheres and lens spaces is considerably richer in topology invariants in comparison to the ensemble of the spatial sections of n-handles [$n = 1, 2, 3$; see, for example, Mandelbaum (1978), Chapter 3], such an approach to the problem of fixing coupling constants (Weinberg, 1989) is more realistic in the case of our ensemble.

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